Ultraviolet cutoff, black-hole-radiation equilibrium, and the big bang

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In the presence of a minimal uncertainty in length, there exists a critical temperature above which the thermodynamics of a gas of radiation changes drastically. We find that the equilibrium temperature of a system composed of a Schwarzschild black hole surrounded by radiation is unaffected by these modifications. This is in agreement with works related to the robustness of the Hawking evaporation. The only change the deformation introduces concerns the critical volume at which the system ceases to be stable. On the contrary, the evolution of the very early universe is sensitive to the new behavior. We readdress the shortcomings of the standard big bang model (flatness, entropy, and horizon problems) in this context, assuming a minimal coupling to general relativity. Although they are not solved, some qualitative differences set in.

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I. INTRODUCTION

The ultimate structure of space time has been the core of many works. Some begin with a fundamental construction such as string theory and find that, when particular fields are turned on, the effective theory can be described as built on a space-time which has modified commutation relations [1–3] or dispersion relations [4]. The same occurs with loop quantum gravity [5]. Another approach consists in using toy models with *ad hoc* modifications in order to study, in a simplified way, the influence of possible departures from usual symmetries at high scale. This last approach has been adopted in the study of the trans-Planckian problem of black-hole evaporation [6–12]. Our work fits in the second approach although the commutators on which we work can be seen as coming from string theory [2]. Even in standard quantum electrodynamics, Planck's radiation law gets corrections due to photon interactions [13]. Here we focus on modifications which may arise from a nontrivial underlying space-time structure.

If one modifies the commutators, one changes the Heisenberg uncertainty relations. The measure on the phase space is no more the same; this results in new partition functions and consequently different thermodynamical behaviors. From the quantum point of view, the energy spectra of systems are modified by the change in the commutation relations.

The influence of this "new" thermodynamics in the early universe has been analyzed in some models with modified dispersion relations [14,15]. However, the equations of states used came only from considerations on bosons. In the unmodified theory this is justified by the fact that the difference between bosons and fermions reduces to a factor 7/8 in the energy densities, pressures, etc. As pointed out in [16], the difference between bosons and fermions in theories with ultraviolet cutoffs is much more pronounced. This naturally raises the question of the way the picture is modified when one considers them altogether. In this work, we provide analytical approximations for the equation of state, the entropy of such a mixture, and we quantify the flatness, entropy, and horizon problem in this framework. Many studies have been devoted to cosmological perturbations in trans-Planckian physics [17–29]; our treatment tackles some of the reasons which led to the inflationary paradigm.

Concerning black holes, it was first realized that, on purely classical grounds, an entropy and a temperature could be associated to these objects [30]. This was confirmed using QFT on a curved background; the exact factor for the temperature was also found [32]. It was then realized that in this derivation, photons were emitted with trans-Planckian frequencies, raising doubts about their treatment as noninteracting particles. It was pointed out afterward that taking somehow into account this effect through dispersion relations which depart from the usual theory at trans-Planckian scale, the Hawking radiation was not intrinsically changed while its derivation became more reliable [6–10].

When the dispersion relation of the photons is the usual one, an equilibrium can be achieved for a system in which a neutral, nonrotating, and noncharged black hole is in a fixed box filled with radiation [33,34]. Moreover, this temperature is the one obtained by Hawking. We analyze how this is affected when nontrivial dispersion relations are considered, in the spirit of [6–10]. Other physical theories such as the ones implying extra dimensions have also consequences on black-hole evolution [31].

The article is organized as follows. In the second section we briefly present a model exhibiting a minimal uncertainty in length and derive its black-body radiation. We find sensible differences between fermions and bosons at very high temperatures [16]. The third section treats a system consisting of a black hole in equilibrium with radiation in the new framework. The last part investigates, quantitatively, how the problems of the standard big bang (flatness, horizon and entropy) are affected.

II. BLACK-BODY RADIATION

A very simple modification of the position-momentum commutation relation leads to a theory possessing a minimal uncertainty in length [35]. Some high-dimensional extensions of this algebra preserve rotational and translational in-

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variance. The model we shall study is given by the following nonvanishing commutators:

$$[\hat{x}_j,\hat{p}_k]=i\hbar(f(\hat{p}^2)\delta_{jk}+g(\hat{p}^2)\hat{p}_j\hat{p}_k),$$

$$g(p^2)=\beta,\quad f(p^2)=\frac{\beta p^2}{-1+\sqrt{1+2\beta p^2}}. \tag{1}$$

This theory has no position representation; the best way to recover information on positions is through the so called quasiposition representation in which the momentum operators read

$$p_k=-i\hbar\sum_{r=0}^{\infty}\left(\frac{\hbar^2\beta}{2}\Delta\right)^r\frac{\partial}{\partial\xi_k},\quad\text{where}\quad\Delta=\sum_{l=1}^{3}\frac{\partial^2}{\partial\xi_l^2}. \tag{2}$$

This representation is found by projecting on states of maximal localization. For details, see [35].

The interest of this toy model is that it displays some features which are thought to appear when quantum gravity takes place. Among them one has nonlocality and the regularization of ultraviolet divergences.

Introducing the momentum scale β, it is straightforward that with the Boltzmann constant k and the light velocity c, one can construct on purely dimensional grounds the characteristic temperature

$$T_c=\frac{c}{k\sqrt{\beta}}. \tag{3}$$

Let us now analyze how radiation gets affected by the new scale. We will use the conventions of [40,41]. Thanks to the deformation of the Klein-Gordon equation [36,37,39,40], the dispersion relation in our case reads

$$E=\frac{c\hbar k}{\left(1+\frac{1}{2}\hbar^2\beta k^2\right)}. \tag{4}$$

The action of the momentum operators [Eq. (2)] on plane waves of wave vectors $\vec{k}$ is finite only if the condition

$$\hbar^2k^2\leqslant\frac{2}{\beta} \tag{5}$$

is satisfied; this is our cutoff. The important quantities are

$$q_{bo}=\sum_{\vec{l}}\log\left[1-\exp\left(-\frac{\epsilon_{\vec{l}}}{kT}\right)\right]=-\log Z_{bo},$$

$$q_{fe}=-\sum_{\vec{l}}\log\left[1+\exp\left(-\frac{\epsilon_{\vec{l}}}{kT}\right)\right]=-\log Z_{fe}, \tag{6}$$

where Z is the grand partition function. The entropy is given by

$$S=-\frac{\partial\Phi}{\partial T},\quad\text{with}\quad\Phi=kT\log Z, \tag{7}$$

while the energy and the particle number read

$$U=\sum_{\vec{l}}\frac{\epsilon_{\vec{l}}}{\exp\left(\frac{\epsilon_{\vec{l}}}{kT}\right)\mp1},\quad N=\sum_{\vec{l}}\frac{1}{\exp\left(\frac{\epsilon_{\vec{l}}}{kT}\right)\mp1}. \tag{8}$$

For bosons, the quantity q linked to the partition function by Eq. (6) can be written as

$$q=4\pi V\left(\frac{kT}{hc}\right)^3\int_0^{\sqrt{2}(T_c/T)}dx\,x^2\times\log\left[1-\exp\left(-\frac{x}{1+\frac{1}{2}\frac{T^2}{T_c^2}x^2}\right)\right], \tag{9}$$

while the energy assumes the following form:

$$U=4\pi V\frac{(kT)^4}{(hc)^3}\int_0^{\sqrt{2}(T_c/T)}dx\frac{x^3}{1+\frac{1}{2}\frac{T^2}{T_c^2}x^2}\times\left[\exp\left(\frac{x}{1+\frac{1}{2}\frac{T^2}{T_c^2}x^2}\right)-1\right]^{-1}. \tag{10}$$

The particle number admits a similar integral expression.

For temperatures greater than or comparable to T_c, the interval of integration is small and a Taylor expansion can be used to obtain an approximation. Working to fourth order, we are led to the following expressions:

$$p_{bo}=\sigma T_c\left[2-\frac{2}{15}\sqrt{2}\frac{T_c}{T}+\sqrt{2}\,\frac{T}{T_c}\left(\frac{8}{3}\log\frac{T}{T_c}+\frac{112}{45}-\frac{4}{3}\log 2\right)\right],$$

$$\rho_{bo}=\sigma T_c\left[-2+\frac{8}{3}\sqrt{2}\frac{T}{T_c}+\frac{4}{15}\sqrt{2}\,\frac{T_c}{T}\right],$$

$$s_{bo}=\sigma\left[\frac{2}{15}\sqrt{2}\left(\frac{T_c}{T}\right)^2+\frac{2}{45}\sqrt{2}\left(60\log\frac{T}{T_c}+116-30\log 2\right)\right],$$

$$N_{bo}=\frac{\sigma}{k}V\left[\frac{1}{3}\,\frac{T_c}{T}-\frac{4}{3}\sqrt{2}+6\frac{T}{T_c}\right],\quad\text{where}\quad\sigma=\pi\frac{k^4T_c^3}{h^3c^3}. \tag{11}$$

The corresponding quantities for fermions (except the pressure) are dominated by constants:

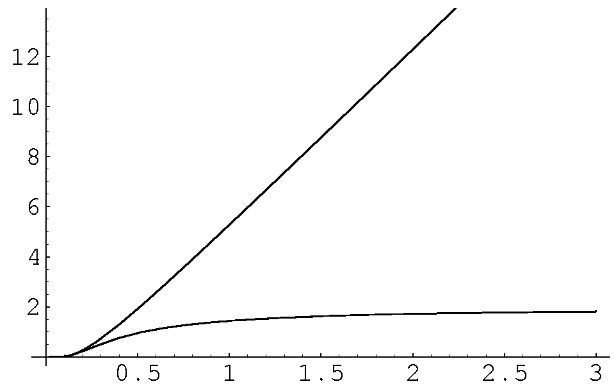

FIG. 1. The energy densities for fermions and bosons are plotted in terms of T/T_c. The unit for energies is $16\pi[(kT_c)^4/(hc)^3]$.

$$p_{fe}=\sigma T_c\left[\frac{2}{5}\sqrt{2}\,\frac{T_c}{T}-2+\frac{8}{3}\log 2\sqrt{2}\,\frac{T}{T_c}\right],$$

$$\rho_{fe}=\sigma T_c\left(2-\frac{4}{5}\sqrt{2}\,\frac{T_c}{T}\right),$$

$$s_{fe}=\sigma\left[\frac{8}{3}\sqrt{2}\,\log 2-\frac{2}{5}\sqrt{2}\left(\frac{T_c}{T}\right)^2\right],$$

$$N_{fe}=\frac{\sigma}{k}V\left(\frac{4}{3}\sqrt{2}-\frac{T_c}{T}\right). \tag{12}$$

The behavior of the energy is depicted in Fig. 1.

At temperatures below T_c, the energy density is polynomial ($\sim T^4$). Above T_c, it becomes linear as obtained in Eq. (11) for bosons while it goes to a constant for fermions, as shown in Eq. (12). The difference between bosons and fermions in the unmodified theory is encoded in the factor 7/8, for the energy contributions, for example. One sees that this is drastically changed here.

III. BLACK-HOLE-RADIATION EQUILIBRIUM

We have seen in the preceding section how thermodynamics is influenced by the existence of a minimum uncertainty in length. We now wish to apply our results to the only systems in which extremely high temperatures can be obtained: black holes and the very early universe.

Using purely classical considerations, it was argued by Beckenstein that the area of a black hole can be interpreted as an entropy while its mass is identified with the energy [30]. The point of view considered in this paper is that the entropy of a black hole comes from a classical reasoning and so is essentially the same as in the unmodified theory. This is linked to the fact that in most phenomenological approaches to trans-Planckian physics, one supposes that the particles evolve on a classical background but are subject to nontrivial dispersion relations for example [6–12].

Let us consider a Schwarzschild black hole surrounded by radiation. The entropy and the energy of such a system read

$$S_{tot}=\frac{4\pi}{l_{pl}^2}M^2+S_{rad}, \quad E_{tot}=Mc^2+E_{rad}, \tag{13}$$

where l_{pl} is the Planck length and we use units in which $k=1$ [38]. If the system is isolated, the total energy is conserved. According to the second law of thermodynamics, equilibrium configurations correspond to maxima of the entropy. Therefore, if the volume of the system is fixed, the derivatives of S_{tot} and E_{tot} vanish at equilibrium. This can be used to obtain a relation between the mass of the black hole and the equilibrium temperature:

$$M=\frac{l_{pl}^2c^2}{8\pi}\left[\frac{dS_{rad}}{dT}\left(\frac{dE_{rad}}{dT}\right)^{-1}\right]_{eq}. \tag{14}$$

In the usual theory, one has

$$\begin{aligned}\frac{dS_{rad}}{dT}\left(\frac{dE_{rad}}{dT}\right)^{-1}&=\frac{d}{dT}\left[\frac{4\pi^2}{45}\,\frac{T^3}{c^3h^3}\left(n_{bo}+\frac{7}{8}n_{fe}\right)V\right]\\&\quad\times\left[\frac{d}{dT}\left(\frac{\pi^2}{15}\,\frac{T^4}{c^3h^3}\left(n_{bo}+\frac{7}{8}n_{fe}\right)V\right)\right]^{-1}\\&=\frac{1}{T}\Rightarrow T_{eq}=\frac{l_{pl}^2c^2}{8\pi}\,\frac{1}{M},\end{aligned} \tag{15}$$

so that the mass of a black hole is inversely proportional to its temperature; this is the Hawking temperature. Including fermions introduces 7/8 factors but does not change the final result.

This temperature is not affected by the presence of a minimal length uncertainty. There is a direct way of obtaining this result. The first law of thermodynamics relates in the following way the changes in the energy E_{rad}, the entropy S_{rad}, and the number of particles N:

$$dE_{rad}=T\,dS_{rad}-p\,dV+\mu\,dN, \tag{16}$$

T being the temperature, p the pressure, and μ the chemical potential. The gas of radiation we consider is contained in a fixed volume ($dV=0$) and has zero chemical potential ($\mu=0$); this gives

$$\frac{dS_{rad}}{dE_{rad}}=\frac{1}{T}, \tag{17}$$

so that the term under square brackets in Eq. (14) takes the value $1/T$. Another derivation, relying on the expressions found in the preceding section, is given in the Appendix.

The equilibrium is stable if the second derivative of the entropy is positive. This gives a particular volume such that the system tends to a black hole surrounded by radiation below it and to pure radiation above it. In our case, this critical volume is

$$V_o=\frac{15c^7h^3l_{pl}^2M^2}{\sqrt{2}T_c^3[(5c^4l_{pl}^4-32\pi^2T_c^2M^2)n_{bo}+96\pi^2T_c^2M^2n_{fe}]}, \tag{18}$$

where again we have put $k=1$. Contrary to the equilibrium temperature which does not feel the modified dispersion relation, the volume which fixes the final fate of the system depends on it. In particular, one sees that it goes like the second power of the black-hole mass, contrary to the usual case where it behaves like its fifth power. Small black holes are the ones which display higher temperatures. The preceding formula is applicable only to such black holes. The coefficients of n_{fe} and n_{bo} are thus positive and V_o never blows up.

IV. THE VERY EARLY UNIVERSE

The influence of trans-Planckian physics on the CMB predictions has been the subject of many studies [17–26]. It was also pointed out that the existence of a new physical scale changes the equation of state for radiation and thus the evolution of the cosmic scale factor [14,15]. To our knowledge, all these studies restricted themselves to the bosonic case. We showed in Sec. II that fermions, compared to bosons have a drastically different behavior above T_c. We will include them to the picture and quantify the horizon, entropy, and flatness problem in this context.

We consider a Robertson-Walker space-time

$$ds^2=-c^2dt^2+a(t)^2\left(\frac{dr^2}{1-Kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right); \tag{19}$$

the constant K can take the values $1,0,-1$, corresponding, respectively, to the closed, flat, and open cases.

To use general relativity one needs to restore covariance somehow. One of the ways to tackle this issue is by introducing a unit vector field u_μ which is timelike and fixes the reference frame in which the modified dispersion relation hold. This field is made dynamical by the incorporation of a Maxwell-type Lagrangian for example. A multiplier is also used to impose the norm of this vector to be equal to 1. In a cosmological solution, this vector depends only on time. For a theory exhibiting a modified dispersion relation of the form $E^2=p^2+ap^4$, a coupling to general relativity has been constructed in which the vector field does not contribute to the energy-momentum tensor while the multiplier vanishes. The details of this procedure can be found in [14]. Such a coupling will be said to be minimal. For simplicity, we shall assume that such a coupling can be constructed in our model. The possibility of more subtle kinetic terms for the vector field is also possible. If one chooses another coupling to gravity our analysis can be seen as giving a heuristic view of the fact that the bosonic and fermionic contents of the universe have, in the presence of an ultraviolet cut off, an influence which is different from the one to which we are accustomed. The same thing holds if ultimately one is not able to build the minimal coupling to gravity we suggested above.

We shall then use the following equations for the evolution of the universe:

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0,\quad \left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3c^2}\rho+\frac{K}{a^2}; \tag{20}$$

the dot means a derivative with respect to time. An important characteristic displayed by the thermodynamics of the system endowed with a minimal uncertainty in length is that the equations of states [see Eqs. (11) and (12)] are not of the type $p=\gamma\rho$, with γ a constant. This class of equations of states is common in cosmology and leads to the scale factor dependence $a\sim t^{(2/3)[1/(1+\gamma)]}$. Therefore, in our case, the derivation of the relation between the scale factor, time, and temperature requires a slightly different treatment. Introducing a prime to denote the derivative with respect to temperature, the first part of Eq. (20), with $K=0$, is transformed into the equation

$$\frac{a'}{a}=-\frac{1}{3}\frac{\rho'(T)}{p(T)+\rho(T)}, \tag{21}$$

which admits a solution by quadratures:

$$a(T)=a_*\exp\left[-\frac{1}{3}\int_{T_*}^{T}d\xi\frac{\rho'(\xi)}{p(\xi)+\rho(\xi)}\right]. \tag{22}$$

As one knows the expressions for density and pressure in terms of the temperature, one can infer the scale factor dependence on temperature. From the second part of Eq. (20), one reads similarly the link between time and temperature:

$$t(T)-t_*=-\frac{1}{3}\sqrt{\frac{3c^2}{8\pi G}}\int_{T_*}^{T}d\xi\frac{\rho'(\xi)}{\sqrt{\rho(\xi)}[p(\xi)+\rho(\xi)]}. \tag{23}$$

In the usual case, one has $p=c^{te}T^4$ and $\rho=3p$. The last two equations then give

$$a_2(T)=\tilde{a}_2\frac{T_{pl}}{T}\quad\text{and}\quad t_2(T)=\tilde{t}_2\left(\frac{T_{pl}}{T}\right)^2+\tilde{t}_3,$$

$$\tilde{t}_2=\frac{3\sqrt{5}}{4\pi\sqrt{\pi}}\frac{1}{\sqrt{n_{bo}+\frac{7}{8}n_{fe}}}t_{pl},$$

$$\text{where}\quad T_{pl}=\frac{1}{k}\sqrt{\frac{c^5\hbar}{G}}\quad\text{and}\quad t_{pl}=\sqrt{\frac{G\hbar}{c^5}} \tag{24}$$

are the Planck temperature and time. From this one finds the usual relation $t\sim 1/T^2$.

The reason of the subscript for the scale factor is the following. The big bang in this model has a radiation period composed of two stages: the first one corresponds to extremely high temperatures and feels the presence of the minimal uncertainty in length. The evolution of the scale factor during that epoch will be denoted by $a_1(t)$ and given in Eq. (26). The second period is the one in which the usual theory becomes valid; its evolution is given by Eq. (24) and its scale factor is denoted by $a_2(t)$.

At very high temperatures, bosons dominate. The evolution of the scale factor in terms of the temperature is then essentially given by the following parametric relations:

$$t\sim\left(\frac{T}{T_c}\right)^{-1/2}\left(\log\frac{T}{T_c}\right)^{-1}, \quad a\sim\left(\log\frac{T}{T_c}\right)^{-1/3}. \tag{25}$$

Compared to the unmodified theory, we find that the scale factor's evolution in terms of temperature is much slower. To put it differently, at the same temperature, the usual big bang would predict a much bigger scale factor, for temperatures well above T_c. The time spent to attain this temperature is also more important in our model.

For future use, we will need the behavior at high temperatures but at an epoch when fermions begin to play a role. Equations (11), (12), (22), (23) lead to the following formulas:

$$a_1(T)=\tilde{a}_1\left[\bar{a}\left(\frac{T}{T_c}\right)^{-2}+2\left(\bar{d}+\bar{c}\log\frac{T}{T_c}\right)\right]^{-1/3},$$

$$t_1(T)=\tilde{t}_1\int_{T/T_c}^{\infty}dx\frac{(\bar{c}x^2-\bar{a})}{\sqrt{x}\sqrt{\bar{c}x^2+\bar{b}x+\bar{a}}\left[\frac{\bar{a}}{2}+(\bar{d}+\bar{c}\log x)x^2\right]},$$

with

$$\tilde{t}_1=\frac{1}{\sqrt{3\pi}}\left(\frac{T_{pl}}{T_c}\right)^2 t_{pl}. \tag{26}$$

The constants $\bar{a}, \ldots, \bar{d}$ embody the dependence of the system on the bosonic-fermionic content at very high temperatures:

$$\bar{a}=\frac{4}{15}\sqrt{2}(n_{bo}^I-3n_{fe}^I), \quad \bar{b}=2(-n_{bo}^I+n_{fe}^I),$$

$$\bar{c}=\frac{8}{3}\sqrt{2}\,n_{bo}^I,$$

$$\bar{d}=\frac{2}{45}\sqrt{2}[116\,n_{bo}^I-30\,(n_{bo}^I-2\,n_{fe}^I)\log 2]. \tag{27}$$

We have used a superscript (n_{bo}^I) to emphasize that the degrees of freedom appearing here are the ones present at $T>T_c$.

The two periods must join smoothly at some point. For illustrative purposes, we shall make the approximation that this occurs at the critical temperature: we shall equalize the scale factors ($a_1=a_2$) and the times ($t_1=t_2$) at this value. The first equation leads to a relation between the constants fixing the scales of the geometries in the two regions:

$$\frac{\tilde{a}_2}{\tilde{a}_1}=(\bar{a}+2\bar{d})\frac{T_c}{T_{pl}}, \tag{28}$$

while the second can be used to extract the value of $\tilde{t}_3$.

A. The flatness/entropy problem

The critical density ρ_{cr} is a fictitious value which gives the same evolution of the scale factor but with $K=0$ in formula (20). Introducing $\Omega=\rho/\rho_{cr}$, one has

$$\Omega-1=\frac{K}{a^2H^2}. \tag{29}$$

In the usual theory, one can use Eq. (24) to show that

$$|\Omega-1|_T=4\left(\frac{\tilde{t}_2}{\tilde{a}_2}\right)^2\left(\frac{T_{pl}}{T}\right)^2, \tag{30}$$

from which one deduces

$$\frac{|\Omega-1|_T}{|\Omega-1|_{T_o}}=\left(\frac{T_o}{T}\right)^2=10^{-64} \quad \text{for} \quad T=T_{pl}. \tag{31}$$

In this formula T_o is the present day temperature of the CMB radiation and T_{pl} is the Planck temperature. As the denominator of the left hand side of the preceding formula is known to be close to unity today, its numerator must have been incredibly close to one at the Planck scale: this is the flatness problem; it can be solved by inflation.

Retaining the dominant contributions in Eq. (26), one obtains for the first part of the radiation period,

$$|\Omega-1|_T=9\,2^{2/3}(\bar{c})^{-1/3}\left(\frac{\tilde{t}_1}{\tilde{a}_1}\right)^2\left(\frac{T}{T_c}\right)^{-1}\left(\log\frac{T}{T_c}\right)^{2/3}, \tag{32}$$

while Eq. (30) is valid for the second period. As we do not have explicitly the scalar factor time dependence, we have obviously used the equation

$$H=\frac{\dot{a}}{a}=\frac{1}{a}\frac{a'}{t'}. \tag{33}$$

Using the matching condition displayed in Eq. (28), one finds the ratio

$$\frac{|\Omega-1|_T}{|\Omega-1|_{T_o}}=Z\left(\frac{T_o}{T_c}\right)^2\left(\frac{T}{T_c}\right)^{-1}\left(\log\frac{T}{T_c}\right)^{2/3},$$

where

$$Z=\frac{4}{15}2^{2/3}(\bar{c})^{-1/3}(\bar{a}+2\bar{d}))^2\left(n_{bo}^{II}+\frac{7}{8}n_{be}^{II}\right). \tag{34}$$

We have used a superscript (n_{bo}^{II}) to emphasize that the particles appearing in this formula are the degrees of freedom present at T_o; their numbers are of course smaller than the ones at the high temperature T which enter into the constants $\bar{a}, \ldots,$ as displayed in Eq. (27).

From the last equation one draws two conclusions. First, as the temperature goes to infinity, the ratio vanishes so that the flatness problem is not solved in this context. This was remarked in a different model [14], relying on numerical computations and in [15] in the presence of a minimal uncertainty in length. The analytical approach followed in our work enables us to say two more things. First, although the flatness problem is not solved, Eq. (34) shows that it is less severe in the presence of a minimal uncertainty in length. Second, the content of the theory in terms of bosons and fermions now plays a role, contrary to the usual theory in which Eq. (31) applies. One also sees that assuming the critical and the Planck temperatures to be different or equal does not matter; above the maximum of the two, our formulas show that the flatness problem remains unsolved.

At very high temperatures, one can rewrite the departure from flatness in terms of the entropy in the following way:

$$|\Omega-1|_T\sim\frac{1}{s^{2/3}\exp\left(\dfrac{3s}{8\sqrt{2}\sigma}\right)}; \tag{35}$$

this differs from the undeformed theory for which one has

$$|\Omega-1|_T\sim\frac{1}{s^{2/3}}. \tag{36}$$

The same conclusion than in the unmodified theory holds: to be almost flat today, the universe had to have a huge entropy density at the initial times.

In our model, one can verify that the total entropy is conserved, order by order. For example, the dominant contribution to the entropy contained in a comoving volume is

$$a^3 s\sim c^{st}\left[\left(\log\frac{T}{T_c}\right)^{-1/3}\right]^3\left(\log\frac{T}{T_c}\right), \tag{37}$$

which is independent of the temperature.

B. The horizon problem

The distance a photon has traveled since the big bang is given by

$$\begin{aligned}R_H&=ca(t_o)\int_0^{t_o}\frac{dt}{a(t)}\\&=-c\,a(t_o)\left[\int_{T_o}^{T_c}t_2'(T)\frac{dT}{a_2(T)}+\int_{T_c}^{\infty}t_1'(T)\frac{dT}{a_1(T)}\right].\end{aligned} \tag{38}$$

The dominant contributions can be written as

$$R_H(T_o)=R_H^*(T_o)\left(1+\gamma\frac{T_o}{T_c}\right), \tag{39}$$

where $R_H^*(T_o)$ is the horizon in the unmodified theory while γ contains the information about the fermionic/bosonic content:

$$\begin{aligned}\gamma&=2^{-2/3}\frac{4\pi}{3\sqrt{15}}\sqrt{n_{bo}^{II}+\frac{7}{8}n_{fe}^{II}(\bar{a}+2\bar{d})}\\&\quad\times\int_1^\infty dx\frac{(\bar{c}x^2-\bar{a})}{x^{7/8}\sqrt{\bar{c}x^2+\bar{b}x+\bar{a}}\left[\dfrac{\bar{a}}{2}+(\bar{d}+\bar{c}\log x)x^2\right]^{2/3}}.\end{aligned} \tag{40}$$

The smallness of today's temperature T_o compared to the critical temperature T_c is such that the correction to the horizon will not be significant. Note, however, that once again the spin content of the universe enters into play in a nontrivial way.

The various quantities which appear in our formulas have the correct behaviors. Considering for example Eq. (40), the term under square root in the integral must obviously be positive. One can rewrite it as a linear combination of n_{fe}^I and n_{bo}^I; the first coefficient vanishes at $x=0.56$ while the second has no zero. This means that the matching temperature cannot be taken to be lower than $0.56T_c$. A precise numerical evaluation of this quantity is possible but one has then to work in a specific model, with n_{fe}^I, n_{bo}^I known. One can nevertheless expect that the departure from usual physics will take place around T_c.

V. CONCLUSIONS

We have studied the thermodynamics induced by a nonlocal theory which exhibits a minimal uncertainty in length. We have obtained that a new behavior sets in at very high temperatures. The difference between fermions and bosons is more important than in the usual case. When studying the equilibrium of a gas of radiation surrounding a black hole, we have suggested a generic argument which assures the universality of the temperature of equilibrium, for reasonable deformations. Our work is in agreement with previous studies which, using the machinery of quantum field theory in curved backgrounds, showed that the Hawking radiation as perceived by a remote observer is not affected when the dispersion relation gets modified at Planckian energies. The only difference we found so far is in the volume which is at the frontier separating the cases in which the final stage contains only radiation from the ones in which the two are present. On the contrary, we have shown that the flatness problem is less severe in this context, contrary to the horizon problem which remains, roughly speaking, untouched. The scale factor and time growths in function of the temperature are slower than as a the usual big bang. We also saw that the entropy per comoving volume was conserved.

One of the important ingredients in our cosmological analysis is the way the theory is coupled to gravity. We assumed, inspired by [14], that there is a "minimal" coupling, i.e., one for which (in the cosmological solution) the unit timelike vector field does not contribute to the energy density. One of the challenges is now to construct such a model explicitly. If one considers other possibilities, quantitative differences are likely to appear. We nevertheless suspect that, qualitatively, the fact that bosons and fermions behave differently at very high temperatures is captured, at least partially, by the treatment we have presented. One of the key issues in a more complete treatment will be the adiabatic expansion of the universe. The problem of the coupling to gravity we assume is that there may exist other gauges in which the unit vector is not timelike. Another prescription would be to postulate that the relations used are valid in the Lorentz frame tied to the cosmic background radiation. The problem then is the loss of covariance.

The commutation relations studied here can be interpreted as phenomenological consequences of string or M theory [2]. Our work, after others, suggests that string cosmology may not be uniquely characterized by the evolution of the fields which appear at the lowest order (such as the dilaton) but also by some nontrivial statistical effects. Finally, the theory exhibiting a minimal length uncertainty may forbid the singularity present in the standard big bang scenario. A similar reasoning was advocated to argue that the Hawking evaporation of a black hole may halt without using the complementarity hypothesis [42].

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APPENDIX: UNIVERSALITY OF HAWKING TEMPERATURE

Let us first consider the regime in which a black hole is in equilibrium with a radiation consisting uniquely of bosons in the extremely high temperatures. Using the formulas displayed in Eq. (11) and retaining only the dominant contributions one has

$$\frac{dS_{rad}}{dT}\left(\frac{dE_{rad}}{dT}\right)^{-1}=\frac{d}{dT}\left(\frac{8}{3}\sqrt{2}\log\frac{T}{T_c}\right)\left[\frac{d}{dT}\left(\frac{8}{3}\sqrt{2}\ \frac{T}{T_c}\right)\right]^{-1}=\frac{1}{T}. \tag{A1}$$

Similarly, one has for a gas containing only fermions,

$$\frac{dS_{rad}}{dT}\left(\frac{dE_{rad}}{dT}\right)^{-1}=\frac{d}{dT}\left(-\frac{2}{5}\sqrt{2}\frac{T_c^2}{T^2}\right)\left[\frac{d}{dT}\left(-\frac{4}{5}\sqrt{2}\ \frac{T_c}{T}\right)\right]^{-1}=\frac{1}{T}, \tag{A2}$$

so that for a mixture the temperature of equilibrium is unchanged.

This relation does not get corrections as one goes beyond the dominant contributions; it is true at all temperatures. This can be seen explicitly for bosons by writing the integral related to the partition function in the following way:

$$q=4\pi V\left(\frac{T_c}{hc}\right)^3\int_0^1 dy\ 2\sqrt{2}y^2 \times\log[1-\exp(g(y))],$$

where

$$g(y)=-\sqrt{2}\ \frac{T_c}{T}\ \frac{y}{1+y^2}. \tag{A3}$$

Contrary to Eq. (9), the temperature does not appear in the upper bound of the integral but in the integrand only. One can carry the derivative with respect to temperature through the integral to obtain the entropy. The energy can in the same way be rewritten as

$$E_{rad}=16\pi V\frac{T_c^4}{(hc)^3}\int_0^1 dy\ \frac{y^3}{1+y^2}\ \frac{1}{[\exp(-g(y))-1]}. \tag{A4}$$

Computing its derivative one finds

$$\frac{dE_{rad}}{dT}=4\pi\sqrt{2}V\frac{T_c^5}{c^3h^3T^2}\int_0^1 \frac{y^4}{(1+y^2)^2}\cosh^2\left(\frac{1}{\sqrt{2}}\ \frac{T_c}{T}\ \frac{y}{1+y^2}\right). \tag{A5}$$

The derivative of the entropy

$$\frac{dS_{rad}}{dT}=-\left(2\frac{\partial q}{\partial T}+T\frac{\partial^2 q}{\partial T^2}\right) \tag{A6}$$

is found to have the same expression, with an extra factor T, so that going back to Eq. (14) the term under square brackets is exactly $1/T$. A similar situation occurs for fermions.

This is in agreement with the idea that the Hawking black-hole temperature measured by an observer at infinity is not affected by a modification of the dispersion relation at trans-Planckian energies [6–10]. In fact, the result we just showed is generic and does not rely very much on the modified dispersion relation. One only needs think about a fixed volume for the black-hole-radiation system and a vanishing chemical potential for the particles comprising the radiation. In fact, using Eqs. (6) and (7) without specifying any dispersion relation, one obtains

$$\frac{dS_{rad}}{dT}=\frac{1}{kT^3}\sum_{\vec{n}}\ \epsilon_{\vec{n}}^2\ \frac{\exp\left(-\frac{\epsilon_{\vec{n}}}{kT}\right)}{\left[1-\exp\left(-\frac{\epsilon_{\vec{n}}}{kT}\right)\right]^2}. \tag{A7}$$

Computing the derivative of the energy given in Eq. (8) leads to a cancellation in Eq. (14) which preserves exactly the last part of Eq. (15). This reasoning is valid for any kind of "reasonable" dispersion relation such that q and its derivatives are finite and the derivation can be carried into the infinite sum.

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